

# KOSTANT'S WEIGHT MULTIPLICITY FORMULA AND THE FIBONACCI NUMBERS

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**ABSTRACT.** It is well known that the dimension of a weight space for a finite dimensional representation of a simple Lie algebra is given by Kostant's weight multiplicity formula. We address the question of how many terms in the alternation contribute to the multiplicity of the zero weight for certain, very special, highest weights. Specifically, we consider the case where the highest weight is equal to the sum of all simple roots. This weight is dominant only in Lie types  $A$  and  $B$ . We prove that in all such cases the number of contributing terms is a Fibonacci number. Combinatorial consequences of this fact are provided.

## 1. INTRODUCTION

Let  $G$  be a simple linear algebraic group over  $\mathbb{C}$ ,  $T$  a maximal algebraic torus in  $G$  of dimension  $r$ , and  $B$ ,  $T \subseteq B \subseteq G$ , a choice of Borel subgroup. Then let  $\mathfrak{g}$ ,  $\mathfrak{h}$ , and  $\mathfrak{b}$  denote the Lie algebras of  $G$ ,  $T$ , and  $B$  respectively. We let  $\Phi$  denote the set of roots corresponding to  $(\mathfrak{g}, \mathfrak{h})$ , and  $\Phi^+ \subseteq \Phi$  is the set of positive roots with respect to  $\mathfrak{b}$ . Let  $\Delta \subseteq \Phi^+$  be the set of simple roots. We denote the set of integral and dominant integral weights by  $P(\mathfrak{g})$  and  $P_+(\mathfrak{g})$  respectively. Let  $W = \text{Norm}_G(T)/T$  denote the Weyl group corresponding to  $G$  and  $T$ , and for any  $w \in W$ , we let  $\ell(w)$  denote the length of  $w$ .

A finite dimensional complex irreducible representation of  $\mathfrak{g}$  is equivalent to a highest weight representation with dominant integral highest weight  $\lambda$ . We denote such a representation by  $L(\lambda)$ . To find the multiplicity of a weight  $\mu$  in  $L(\lambda)$ , we use Kostant's weight multiplicity formula, [5]:

$$(1.1) \quad m(\lambda, \mu) = \sum_{\sigma \in W} (-1)^{\ell(\sigma)} \wp(\sigma(\lambda + \rho) - (\mu + \rho)),$$

where  $\wp$  denotes Kostant's partition function and  $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$ . Recall that Kostant's partition function is the non-negative integer valued function,  $\wp$ , defined on  $\mathfrak{h}^*$ , by  $\wp(\xi) =$  number of ways  $\xi$  may be written as a non-negative integral sum of positive roots, for  $\xi \in \mathfrak{h}^*$ .

With the aim of describing the contributing terms of (1.1) we introduce the following.

**Definition 1.1.** For  $\lambda, \mu$  dominant integral weights of  $\mathfrak{g}$  define the Weyl alternation set to be

$$\mathcal{A}(\lambda, \mu) = \{\sigma \in W \mid \wp(\sigma(\lambda + \rho) - (\mu + \rho)) > 0\}.$$

Therefore,  $\sigma \in \mathcal{A}(\lambda, \mu)$  if and only if  $\sigma(\lambda + \rho) - (\mu + \rho)$  can be written as a nonnegative integral combination of positive roots.

In [3], we considered the adjoint representation of  $\mathfrak{sl}_{r+1} = \mathfrak{sl}_{r+1}(\mathbb{C})$ , whose highest weight is the highest root, which is defined as the sum of the simple roots. In this case we proved:

**Theorem.** *If  $r \geq 1$  and  $\tilde{\alpha}$  is the highest root of  $\mathfrak{sl}_{r+1}$ , then  $|\mathcal{A}(\tilde{\alpha}, 0)| = F_r$ , where  $F_r$  denotes the  $r^{\text{th}}$  Fibonacci number.*

In this note we specialize to  $\mathfrak{so}_{2r+1} = \mathfrak{so}_{2r+1}(\mathbb{C})$  and, in Section 2, prove:

**Theorem 1.1.** *If  $r \geq 2$  and  $\varpi_1 = \sum_{\alpha \in \Delta} \alpha$  is a fundamental weight of  $\mathfrak{so}_{2r+1}$ , then  $|\mathcal{A}(\varpi_1, 0)| = F_{r+1}$ , where  $F_{r+1}$  denotes the  $(r+1)^{\text{th}}$  Fibonacci number.*

This result gives rise to some combinatorial identities associated to a Cartan subalgebra of  $\mathfrak{so}_{2r+1}$ , which we present in Section 3. The non-zero weights,  $\mu$ , of  $\mathfrak{so}_{2r+1}$  are considered in Section 4 from the same point of view.

In Sections 5 and 6, we prove that the weight defined by the sum of the simple roots is not a dominant integral weight of the Lie algebras  $\mathfrak{sp}_{2r}(\mathbb{C})$  (for  $r \geq 3$ ),  $\mathfrak{so}_{2r}(\mathbb{C})$  (for  $r \geq 4$ ),  $G_2$ ,  $F_4$ ,  $E_6$ ,  $E_7$ , and  $E_8$ , and hence does not correspond to finite-dimensional representation.

## 2. THE ZERO WEIGHT SPACE OF $\mathfrak{so}_{2r+1}$

Let  $r \geq 2$ , and let  $G = SO_{2r+1}(\mathbb{C}) = \{g \in SL_{2r+1}(\mathbb{C}) : g^t = g^{-1}\}$  be the special orthogonal group of  $(2r+1) \times (2r+1)$  matrices over  $\mathbb{C}$ . Let  $\mathfrak{g} = \mathfrak{so}_{2r+1}(\mathbb{C}) = \{X \in M_{2r+1}(\mathbb{C}) : X^t = -X\}$ , and<sup>1</sup>  $\mathfrak{h} = \{\text{diag}[a_1, \dots, a_r, 0, -a_r, \dots, -a_1] \mid a_1, \dots, a_r \in \mathbb{C}\}$  be a fixed choice of Cartan subalgebra. For  $1 \leq i \leq r$ , define the linear functionals  $\varepsilon_i : \mathfrak{h} \rightarrow \mathbb{C}$  by  $\varepsilon_i(H) = a_i$ , for any  $H = \text{diag}[a_1, \dots, a_r, 0, -a_r, \dots, -a_1] \in \mathfrak{h}$ .

For each  $1 \leq i \leq r-1$ , let  $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$  and let  $\alpha_r = \varepsilon_r$ . Then the set of simple and positive roots are  $\Delta = \{\alpha_1, \dots, \alpha_r\}$  and  $\Phi^+ = \{\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j : 1 \leq i < j \leq r\} \cup \{\varepsilon_i : 1 \leq i \leq r\}$ , respectively. The fundamental weights are defined by  $\varpi_i = \varepsilon_1 + \dots + \varepsilon_i$  for  $1 \leq i \leq r-1$ , and  $\varpi_r = \frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \dots + \varepsilon_r)$ . Then observe that  $\varpi_1 = \varepsilon_1 = \alpha_1 + \dots + \alpha_r$ , and  $\rho = (r - \frac{1}{2})\varepsilon_1 + (r - \frac{3}{2})\varepsilon_2 + \dots + \frac{3}{2}\varepsilon_{r-1} + \frac{1}{2}\varepsilon_r$ . Let  $(,)$  denote the symmetric bilinear form on  $\mathfrak{h}^*$  corresponding to the trace form as in [2].

For  $1 \leq i \leq r-1$ ,  $s_{\alpha_i}(\varepsilon_k) = \varepsilon_{\sigma(k)}$ , where  $\sigma$  is the transposition  $(i \ i+1) \in S_r$ , and  $s_{\alpha_r}(\varepsilon_k) = \varepsilon_k$  provided  $k \neq r$  and  $s_{\alpha_r}(\varepsilon_r) = -\varepsilon_r$ . For any  $1 \leq i \leq r$ , let  $s_i := s_{\alpha_i}$ . Then the Weyl group,  $W$ , acts on  $\mathfrak{h}^*$  by signed permutations of  $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_r$  and is generated by the simple root reflections  $s_1, \dots, s_r$ .

**Proposition 2.1.** *Let  $\sigma = s_{i_1} s_{i_2} \dots s_{i_k} \in W$ , for some nonconsecutive integers  $i_1, i_2, \dots, i_k$  between 2 and  $r$ . Then  $\sigma(\varpi_1 + \rho) - \rho = \varpi_1 - \sum_{j=1}^k \alpha_{i_j}$  is a nonnegative integral combination of positive roots.*

*Proof.* Recall that for any  $1 \leq i, j \leq r$ ,  $s_i(\varpi_j) = \varpi_j - \delta_{ij} \alpha_j$ , where  $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$  Also, by definition,  $s_i(\alpha_j) = \alpha_j - \frac{2(\alpha_j, \alpha_i)}{(\alpha_i, \alpha_i)} \alpha_i$ , for any  $1 \leq i \leq r$ . Observe that  $(\alpha_i, \alpha_j) = 0$ , whenever

<sup>1</sup> $\text{diag}[a_1, \dots, a_n]$  denotes a diagonal  $n \times n$  matrix with entries  $a_1, \dots, a_n$ .

$i, j$  are nonconsecutive integers between 1 and  $r$ . Thus, given  $\sigma = s_{i_1}s_{i_2}\cdots s_{i_k} \in W$ , with  $i_1, i_2, \dots, i_k$  nonconsecutive integers between 2 and  $r$ , we have that

$$\begin{aligned}\sigma(\varpi_1 + \rho) - \rho &= s_{i_1}s_{i_2}\cdots s_{i_k}(\varpi_1 + (\varpi_1 + \cdots + \varpi_r)) - \rho \\ &= 2\varpi_1 + \varpi_2 + \cdots + \varpi_r - (\alpha_{i_1} + \cdots + \alpha_{i_k}) - \rho \\ &= \varpi_1 - \sum_{j=1}^{j=k} \alpha_{i_j}.\end{aligned}$$

Now since  $\varpi_1 = \alpha_1 + \cdots + \alpha_r$ , notice  $\varpi_1 - \sum_{j=1}^{j=k} \alpha_{i_j}$  is a nonnegative integral combination of positive roots.  $\square$

**Theorem 2.1.** *Let  $\sigma \in W$ . Then  $\sigma \in \mathcal{A}(\varpi_1, 0)$  if and only if  $\sigma = s_{i_1}s_{i_2}\cdots s_{i_k}$ , for some nonconsecutive integers  $i_1, \dots, i_k$  between 2 and  $r$ .*

*Proof.* ( $\Rightarrow$ ) Let  $\sigma \in \mathcal{A}(\varpi_1, 0)$ , and proceed by induction on  $\ell(\sigma)$ . If  $\ell(\sigma) = 0$ , then  $\sigma = 1$ . If  $\sigma = s_i$ , for some  $1 \leq i \leq r$ , then  $s_i(\varpi_1 + \rho) - \rho = s_i(2\varpi_1 + \varpi_2 + \cdots + \varpi_r) - (\varpi_1 + \cdots + \varpi_r)$   

$$= \begin{cases} -\alpha_1 + \alpha_2 + \cdots + \alpha_r & \text{if } i = 1 \\ \alpha_1 + \cdots + \alpha_{i-1} + \alpha_{i+1} + \cdots + \alpha_r & \text{otherwise.} \end{cases}$$
If  $i = 1$ , then we get a contradiction to  $\sigma \in \mathcal{A}(\varpi_1, 0)$ . Thus  $\sigma = s_i$ , where  $2 \leq i \leq r$ . If  $\ell(\sigma) = 2$ , then  $\sigma = s_i s_j$  for some distinct integers  $1 \leq i, j \leq r$ . Clearly  $i, j \neq 1$ , else  $\sigma \notin \mathcal{A}(\varpi_1, 0)$ . Now notice

$$\begin{aligned}s_i s_j(\varpi_i + \rho) - \rho &= \varpi_1 + \rho - \alpha_i - (\alpha_j - \frac{2(\alpha_i, \alpha_j)}{(\alpha_1, \alpha_i)} \alpha_i) - \rho \\ &= \varpi_1 - \alpha_i - \alpha_j + \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)} \alpha_i.\end{aligned}$$

If  $i$  and  $j$  are consecutive integers between 2 and  $r$ , then  $j = i \pm 1$  and, in either case,  $(\alpha_i, \alpha_j) = -1$ . Hence  $s_i s_j \notin \mathcal{A}(\varpi_1, 0)$ , a contradiction. Thus  $i$  and  $j$  are nonconsecutive integers between 2 and  $r$ .

Now assume that for any  $\sigma \in \mathcal{A}(\varpi_1, 0)$  with  $\ell(\sigma) := n \leq k$ , there exist nonconsecutive integers  $i_1, \dots, i_n$  between 2 and  $r$  such that  $\sigma = s_{i_1}s_{i_2}\cdots s_{i_n}$ . Let  $\sigma \in \mathcal{A}(\varpi_1, 0)$ , such that  $\ell(\sigma) = k + 1$ . Hence there exist distinct integers  $i_1, i_2, \dots, i_{k+1}$  between 1 and  $r$  such that  $\sigma = s_{i_1}s_{i_2}\cdots s_{i_{k+1}}$ . Let  $\sigma = s_{i_1}\pi$ , where  $\pi = s_{i_2}\cdots s_{i_{k+1}}$ , with  $\ell(\pi) = k$ . Since  $\sigma \in \mathcal{A}(\varpi_1, 0)$ , we have that  $\pi \in \mathcal{A}(\varpi_1, 0)$ , and by induction hypothesis,  $i_2, \dots, i_{k+1}$  are nonconsecutive integers between 2 and  $r$ . By Proposition 2.1,  $\pi(\varpi_1 + \rho) = \varpi_1 + \rho - \sum_{j=2}^{j=k+1} \alpha_{i_j}$ , and hence

$$\begin{aligned}\sigma(\varpi_1 + \rho) - \rho &= s_{i_1}\pi(\varpi_1 + \rho) - \rho = s_{i_1}(\varpi_1 + \rho - \sum_{j=2}^{j=k+1} \alpha_{i_j}) - \rho \\ &= \varpi_1 + \rho - \alpha_{i_1} - \sum_{j=2}^{j=k+1} (\alpha_{i_j} - \frac{2(\alpha_{i_j}, \alpha_{i_1})}{(\alpha_{i_1}, \alpha_{i_1})} \alpha_{i_1}) - \rho \\ &= \varpi_1 - \sum_{j=1}^{j=k+1} \alpha_{i_j} + 2 \sum_{j=2}^{j=k+1} \frac{(\alpha_{i_j}, \alpha_{i_1})}{(\alpha_{i_1}, \alpha_{i_1})} \alpha_{i_1}.\end{aligned}$$

Observe that if  $i_1 = 1$ , then  $\sigma(\varpi_1 + \rho) - \rho \notin \mathcal{A}(\varpi_1, 0)$ , a contradiction. Now suppose there exists  $2 \leq j \leq r$  such that  $i_j$  and  $i_1$  are consecutive integers. Then  $(\alpha_{i_j}, \alpha_{i_1}) = -1$ ,

and hence the coefficient of  $\alpha_{i_1}$  is negative. Thus  $\sigma \notin \mathcal{A}(\varpi_1, 0)$ , a contradiction. Therefore  $i_1, i_2, \dots, i_{k+1}$  are nonconsecutive integers between 2 and  $r$ .

( $\Leftarrow$ ) By Proposition 2.1, if  $\sigma = s_{i_1} s_{i_2} \cdots s_{i_k} \in W$ , for some nonconsecutive integers  $i_1, \dots, i_k$  between 2 and  $r$ , then  $\sigma(\varpi_1 + \rho) - \rho$  is a nonnegative integral combination of positive roots and hence  $\sigma \in \mathcal{A}(\varpi_1, 0)$ .  $\square$

**Definition 2.1.** The Fibonacci numbers are defined, in [7], by the recurrence relation  $F_1 = F_2 = 1$ , and  $F_n = F_{n-1} + F_{n-2}$ , for  $n \geq 2$ .

We now give the following:

*Proof of Theorem 1.1.* By Theorem 2.1, we know  $\mathcal{A}(\varpi_1, 0) = \{\sigma \in W \mid \sigma = s_{i_1} \cdots s_{i_k}, \text{ for some nonconsecutive integers } 2 \leq i_1, \dots, i_k \leq r\}$ . An induction argument shows that for any  $n \geq 1$ , the number of sequences consisting of nonconsecutive integers between 1 and  $n$  is given by  $F_{n+2}$ . Thus  $|\mathcal{A}(\varpi_1, 0)| = F_{r+1}$ .  $\square$

### 3. A $q$ -ANALOG

The  $q$ -analog of Kostant's partition function is the polynomial valued function,  $\wp_q$ , defined on  $\mathfrak{h}^*$  by  $\wp_q(\xi) = c_0 + c_1 q + \cdots + c_k q^k$ , where  $c_j$  = number of ways to write  $\xi$  as a non-negative integral sum of exactly  $j$  positive roots, for  $\xi \in \mathfrak{h}^*$ . The  $q$ -analog of Kostant's weight multiplicity formula is defined, in [6], as:

$$m_q(\lambda, \mu) = \sum_{\sigma \in W} (-1)^{\ell(\sigma)} \wp_q(\sigma(\lambda + \rho) - (\mu + \rho)).$$

It is known that the multiplicity of the zero weight in the representation  $L(\varpi_1)$  is equal to 1, see [1]. In this section, we give a combinatorial proof of this fact, by proving the following.

**Theorem 3.1.** Let  $r \geq 2$  and let  $\varpi_1 = \sum_{\alpha \in \Delta} \alpha$  be a fundamental weight of  $\mathfrak{so}_{2r+1}$ . Then  $m_q(\varpi_1, 0) = q^r$ .

Observe that the subset of positive roots of  $\mathfrak{so}_{2r+1}$  used to write  $\sigma(\varpi_1 + \rho) - \rho$ , for any  $\sigma \in \mathcal{A}(\varpi_1, 0)$ , is equal to the set of positive roots of  $\mathfrak{sl}_{r+1}$ . Therefore, the following lemmas and propositions follow from Lemma 3.1 and Proposition 3.2 in [3].

**Lemma 3.1.** The cardinality of the sets  $\{\sigma \in \mathcal{A}(\varpi_1, 0) : \ell(\sigma) = k \text{ and } \sigma \text{ contains no } s_r \text{ factor}\}$  and  $\{\sigma \in \mathcal{A}(\varpi_1, 0) : \ell(\sigma) = k \text{ and } \sigma \text{ contains an } s_r \text{ factor}\}$  are  $\binom{r-1-k}{k}$  and  $\binom{r-2-k}{k}$ , respectively. Also  $\max\{\ell(\sigma) : \sigma \in \mathcal{A}(\varpi_1, 0) \text{ and } \sigma \text{ contains no } s_r \text{ factor}\} = \lfloor \frac{r-1}{2} \rfloor$  and  $\max\{\ell(\sigma) : \sigma \in \mathcal{A}(\varpi_1, 0) \text{ and } \sigma \text{ contains an } s_r \text{ factor}\} = \lfloor \frac{r-2}{2} \rfloor$ .

**Proposition 3.1.** Let  $\sigma \in \mathcal{A}(\varpi_1, 0)$ . Then

$$\wp_q(\sigma(\varpi_1 + \rho) - \rho) = \begin{cases} q^{1+\ell(\sigma)}(1+q)^{r-1-2\ell(\sigma)} & \text{if } \sigma \text{ contains no } s_r \text{ factor} \\ q^{1+\ell(\sigma)}(1+q)^{r-2-2\ell(\sigma)} & \text{if } \sigma \text{ contains an } s_r \text{ factor.} \end{cases}$$

Now can now prove the closed formula for the  $q$ -multiplicity of the zero weight in  $L(\varpi_1)$ .

*Proof of Theorem 3.1.* Observe that

$$m_q(\varpi_1, 0) = \sum_{\substack{\sigma \in \mathcal{A}(\varpi_1, 0) \\ \text{with no } s_r \text{ factor}}} (-1)^{\ell(\sigma)} \wp_q(\sigma(\varpi_1 + \rho) - \rho) + \sum_{\substack{\sigma \in \mathcal{A}(\varpi_1, 0) \\ \text{with an } s_r \text{ factor}}} (-1)^{\ell(\sigma)} \wp_q(\sigma(\varpi_1 + \rho) - \rho).$$

By Lemma 3.1, Proposition 3.1 and Proposition 3.3 in [3] it follows that

$$\begin{aligned}
\sum_{\substack{\sigma \in \mathcal{A}(\varpi_1, 0) \\ \text{with no } s_r \text{ factor}}} (-1)^{\ell(\sigma)} \wp_q(\sigma(\varpi_1 + \rho) - \rho) &= \sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor} (-1)^k \binom{r-1-k}{k} q^{1+k} (1+q)^{r-1-2k} \\
&= \sum_{i=1}^r q^i, \text{ and} \\
\sum_{\substack{\sigma \in \mathcal{A}(\varpi_1, 0) \\ \text{with an } s_r \text{ factor}}} (-1)^{\ell(\sigma)} \wp_q(\sigma(\varpi_1 + \rho) - \rho) &= \sum_{k=0}^{\lfloor \frac{r-2}{2} \rfloor} (-1)^{1+k} \binom{r-2-k}{k} q^{1+k} (1+q)^{r-2-2k} \\
&= - \sum_{i=1}^{r-1} q^i.
\end{aligned}$$

Therefore,  $m_q(\varpi_1, 0) = (q + q^2 + \cdots + q^{r-1} + q^r) - (q + q^2 + \cdots + q^{r-1}) = q^r$ .  $\square$

**Corollary 3.1.** *Let  $r \geq 2$  and let  $\varpi_1 = \sum_{\alpha \in \Delta} \alpha$  be a fundamental weight of  $\mathfrak{so}_{2r+1}$ . Then  $m(\varpi_1, 0) = 1$ .*

*Proof.* Follows directly from Theorem 3.1, since  $m(\varpi_1, 0) = m_q(\varpi_1, 0)|_{q=1} = 1$ .  $\square$

#### 4. NON-ZERO WEIGHT SPACES OF $\mathfrak{so}_{2r+1}$

We now consider the non-zero dominant weights,  $\mu$ , of  $\mathfrak{so}_{2r+1}$  and compute the Weyl alternation sets  $\mathcal{A}(\varpi_1, \mu)$ . Throughout this section  $r \geq 2$ .

**Theorem 4.1.** *If  $\mu \in P_+(\mathfrak{so}_{2r+1})$  and  $\mu \neq 0$ , then  $\mathcal{A}(\varpi_1, \mu) = \begin{cases} \{1\} & \text{if } \mu = \varpi_1 \\ \emptyset & \text{otherwise.} \end{cases}$*

We begin by proving the following Propositions, from which Theorem 4.1 follows.

**Proposition 4.1.** *If  $\varpi_1 = \sum_{\alpha \in \Delta} \alpha$  is a fundamental weight of  $\mathfrak{so}_{2r+1}$ , then  $\mathcal{A}(\varpi_1, \varpi_1) = \{1\}$ .*

*Proof.* Since  $\varpi_1 = \alpha_1 + \cdots + \alpha_r$ , notice  $\sigma(\varpi_1 + \rho) - \rho - \varpi_1$  is a non-negative integral sum of positive roots only if  $\sigma(\varpi_1 + \rho) - \rho$  is. By Theorem 2.1 we know  $\sigma(\varpi_1 + \rho) - \rho$  is a non-negative integral sum of positive roots if and only if  $\sigma = s_{i_1} s_{i_2} \cdots s_{i_k}$ , for some nonconsecutive integers  $i_1, \dots, i_k$  between 2 and  $r$ . Hence  $\mathcal{A}(\varpi_1, \varpi_1) \subset \mathcal{A}(\varpi_1, 0)$ . Suppose that  $\sigma \in \mathcal{A}(\varpi_1, \varpi_1)$  with  $\ell(\sigma) = k \geq 1$ , then there exist nonconsecutive integers  $i_1, \dots, i_k$  between 2 and  $r$  such that  $\sigma = s_{i_1} s_{i_2} \cdots s_{i_k}$ . By Proposition 2.1 we have that  $\sigma(\varpi_1 + \rho) - \rho = \varpi_1 - \sum_{j=1}^k \alpha_{i_j}$ . Then notice  $\sigma(\varpi_1 + \rho) - \rho - \varpi_1$  will not be a non-negative integral sum of positive roots, reaching a contradiction. Thus  $\ell(\sigma) = 0$  and  $\sigma = 1$ .  $\square$

**Proposition 4.2.** *Let  $\mu \in P_+(\mathfrak{so}_{2r+1})$ , and  $\mu \neq 0$ . Then there exists  $\sigma \in W$  such that  $\wp(\sigma(\varpi_1 + \rho) - \rho - \mu) > 0$  if and only if  $\mu = \varpi_1$ .*

*Proof.* ( $\Rightarrow$ ) Let  $\mu \in P_+(\mathfrak{so}_{2r+1})$  with  $\mu \neq 0$ , and assume  $\sigma \in W$  such that  $\wp(\sigma(\varpi_1 + \rho) - \rho - \mu) > 0$ . By Proposition 3.1.20 in [2], we know that  $P_+(\mathfrak{so}_{2r+1})$  consists of all weights  $\mu = k_1 \varepsilon_1 + k_2 \varepsilon_2 + \cdots + k_r \varepsilon_r$ , with  $k_1 \geq k_2 \geq \cdots \geq k_r \geq 0$ . Here  $2k_i$ , and  $k_i - k_j$  are integers for all  $i, j$ .

Now observe that  $\sigma(\varpi_1 + \rho) - \rho - \mu = \sigma((r + \frac{1}{2})\varepsilon_1 + (r - \frac{3}{2})\varepsilon_2 + (r - \frac{5}{2})\varepsilon_3 + \cdots + \frac{1}{2}\varepsilon_{r-1} + \varepsilon_r) - ((r - \frac{1}{2})\varepsilon_1 + (r - \frac{3}{2})\varepsilon_2 + \cdots + \frac{1}{2}\varepsilon_r) - (k_1\varepsilon_1 + \cdots + k_r\varepsilon_r)$ . Let  $a_i$  denote the coefficient

$$\text{of } \alpha_i \text{ in } \sigma(\varpi_1 + \rho) - \rho - \mu. \text{ Then } a_1 = \begin{cases} -i + 1 - k_1 & \text{if } \sigma(\varepsilon_1) = \varepsilon_i \text{ for } 2 \leq i \leq r \\ -2r + i - k_1 & \text{if } \sigma(\varepsilon_1) = -\varepsilon_i \text{ for } 2 \leq i \leq r \\ 1 - k_1 & \text{if } \sigma(\varepsilon_1) = \varepsilon_1 \\ -2r - k_1 & \text{if } \sigma(\varepsilon_1) = -\varepsilon_1. \end{cases}$$

Since  $r \geq 2$  and  $a_1 \in \mathbb{N}$ , we have that  $\sigma(\varepsilon_1) = \varepsilon_1$  and  $a_1 = 1 - k_1$ . If  $k_1 = 0$ , then  $k_i = 0$  for all  $1 \leq i \leq r$ , and so  $\mu = 0$ , a contradiction. Hence  $k_1 = 1$ . Since  $k_i - k_j \in \mathbb{Z}$  for all  $i$  and  $j$ , and since  $1 = k_1 \geq k_2 \geq k_3 \geq \cdots \geq k_r \geq 0$ , we have that  $k_i = 0$  or  $1$ , for all  $2 \leq i \leq r$ . We want to show that  $k_i = 0$  for all  $2 \leq i \leq r$ . It suffices to show  $k_2 = 0$ . A simple computation

$$\text{shows that } a_2 = \begin{cases} -i + 2 - k_2 & \text{if } \sigma(\varepsilon_2) = \varepsilon_i \text{ for } 3 \leq i \leq r \\ -2r + i + 1 - k_2 & \text{if } \sigma(\varepsilon_2) = -\varepsilon_i \text{ for } 3 \leq i \leq r \\ -k_2 & \text{if } \sigma(\varepsilon_2) = \varepsilon_2 \\ -2r + 3 - k_2 & \text{if } \sigma(\varepsilon_2) = -\varepsilon_2. \end{cases}$$

Since  $r \geq 2$  and  $a_2 \in \mathbb{N}$ , we have that  $\sigma(\varepsilon_2) = \varepsilon_2$ , and hence  $k_2 = 0$ . Thus  $\mu = \varepsilon_1 = \varpi_1$ .

( $\Leftarrow$ ) By Proposition 4.1, we know if  $\mu = \varpi_1$ , then  $\wp(\sigma(\varpi_1 + \rho) - \rho - \varpi_1) > 0$  when  $\sigma = 1$ .  $\square$

**Theorem 4.2.** *If  $\mu \in P(\mathfrak{so}_{2r+1})$ , then  $m(\varpi_1, \mu) = \begin{cases} 1 & \text{if } \mu = 0 \text{ or } \mu \in W \cdot \varpi_1 \\ \emptyset & \text{otherwise.} \end{cases}$*

*Proof.* Recall that given  $\mu \in P(\mathfrak{so}_{2r+1})$ , there exists  $w \in W$  and  $\xi \in P_+(\mathfrak{so}_{2r+1})$  such that  $w(\xi) = \mu$  and also recall that weight multiplicities are invariant under  $W$  (Propositions 3.1.20, 3.2.27 in [2]). Thus it suffices to consider  $\mu \in P_+(\mathfrak{so}_{2r+1})$ . Corollary 3.1 gives  $m(\tilde{\alpha}, 0) = 1$ , while Theorem 4.1 implies  $m(\varpi_1, \varpi_1) = 1$  and  $m(\varpi_1, \mu) = 0$ , whenever  $\mu \in P_+(\mathfrak{so}_{2r+1}) \setminus \{0, \varpi_1\}$ .  $\square$

## 5. THE CLASSICAL LIE ALGEBRAS $\mathfrak{sp}_{2r}$ AND $\mathfrak{so}_{2r}$

In this section we consider the classical Lie algebras  $\mathfrak{sp}_{2r}(\mathbb{C})$  and  $\mathfrak{so}_{2r}(\mathbb{C})$  and prove.

**Theorem 5.1.** *If  $\mathfrak{g}$  is the classical Lie algebra  $\mathfrak{sp}_{2r}(\mathbb{C})$  (with  $r \geq 3$ ) or  $\mathfrak{so}_{2r}(\mathbb{C})$  (with  $r \geq 4$ ) and  $\Delta$  denotes a set of simple roots, then the weight  $\sum_{\alpha \in \Delta} \alpha$  is not a dominant integral weight of  $\mathfrak{g}$ .*

*Proof.* We follow the notation in [2]. In the case of  $\mathfrak{sp}_{2r} = \mathfrak{sp}_{2r}(\mathbb{C})$ , with  $r \geq 3$ , for each  $1 \leq i \leq r - 1$ , let  $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$  and let  $\alpha_r = 2\varepsilon_r$ . Then the set of simple roots is  $\Delta = \{\alpha_1, \dots, \alpha_r\}$ . The fundamental weights are defined by  $\varpi_i = \varepsilon_1 + \cdots + \varepsilon_i$ , for  $1 \leq i \leq r$ . Now notice  $\alpha_1 + \cdots + \alpha_r = \varepsilon_1 + \varepsilon_r = \varpi_1 - \varpi_{r-1} + \varpi_r$ . Thus, the weight defined by the sum of the simple roots is not a dominant weight of  $\mathfrak{sp}_{2r}$ , for  $r \geq 3$ .

In the case  $\mathfrak{so}_{2r} = \mathfrak{so}_{2r}(\mathbb{C})$ , with  $r \geq 4$ , for each  $1 \leq i \leq r - 1$ , let  $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$  and let  $\alpha_r = \varepsilon_{r-1} + \varepsilon_r$ . Then the set of simple roots is  $\Delta = \{\alpha_1, \dots, \alpha_r\}$ . The fundamental weights are defined by  $\varpi_i = \varepsilon_1 + \cdots + \varepsilon_i$  if  $1 \leq i \leq r - 1$ , and  $\varpi_{r-1} = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{r-1} - \varepsilon_r)$  and  $\varpi_r = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{r-1} + \varepsilon_r)$ . Now notice  $\alpha_1 + \cdots + \alpha_r = \varepsilon_1 + \varepsilon_{r-1} = \varpi_1 - \varpi_{r-2} + \varpi_{r-1} + \varpi_r$ .

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${}^2\mathbb{N} = \{0, 1, 2, 3, \dots\}$ .

Thus, the weight defined by the sum of the simple roots is not a dominant weight of  $\mathfrak{so}_{2r}$ , for  $r \geq 4$ .  $\square$

## 6. EXCEPTIONAL LIE ALGEBRAS

In this section we consider the exceptional simple Lie algebras over  $\mathbb{C}$  and prove.

**Theorem 6.1.** *If  $\mathfrak{g}$  is an exceptional simple Lie algebra of type  $G_2$ ,  $F_4$ ,  $E_6$ ,  $E_7$ , or  $E_8$  and  $\Delta$  denotes a set of simple roots, then the weight  $\sum_{\alpha \in \Delta} \alpha$  is not a dominant integral weight of  $\mathfrak{g}$ .*

*Proof.* In each case we will describe, as in [4], the underlying vector space  $V$  and the root system  $\Phi$  as a subset of  $V$ . In each case the root system will be a subspace of some  $\mathbb{R}^k = \{\sum_{i=1}^k a_i e_i\}$ , where  $\{e_i : 1 \leq i \leq k\}$  is the standard orthonormal basis and the  $a_i$ 's are real numbers.

The underlying vector space of the exceptional Lie algebra  $G_2$  is  $V = \{v \in \mathbb{R}^3 | (v, e_1 + e_2 + e_3) = 0\}$ , and the root system is given by

$$\Phi = \{\pm(e_1 - e_2), \pm(e_2 - e_3), \pm(e_1 - e_3)\} \cup \{\pm(2e_1 - e_2 - e_3), \pm(2e_2 - e_1 - e_3), \pm(2e_3 - e_1 - e_2)\}.$$

The set of simple roots is  $\Delta = \{\alpha_1, \alpha_2\}$ , where  $\alpha_1 = e_1 - e_2$  and  $\alpha_2 = -2e_1 + e_2 + e_3$ , and the fundamental weights, in terms of the simple roots, are  $\varpi_1 = 2\alpha_1 + \alpha_2$  and  $\varpi_2 = 3\alpha_1 + \alpha_2$ . Observe that  $\alpha_1 + \alpha_2 = -\varpi_1 + \varpi_2$ , and hence not a dominant integral weight of  $G_2$ .

The underlying vector space of the exceptional Lie algebra  $F_4$  is  $V = \mathbb{R}^4$ , and the root system is given by  $\Phi = \{\pm e_i \pm e_j | i < j\} \cup \{\pm e_i\} \cup \{\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)\}$ . The set of simple roots is  $\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ , where  $\alpha_1 = \frac{1}{2}(e_1 - e_2 - e_3 - e_4)$ ,  $\alpha_2 = e_4$ ,  $\alpha_3 = e_3 - e_4$ , and  $\alpha_4 = e_2 - e_3$ . The fundamental weights, in terms of the simple roots, are

$$\begin{aligned} \varpi_1 &= 2\alpha_1 + 3\alpha_2 + 2\alpha_3 + \alpha_4 \\ \varpi_2 &= 3\alpha_1 + 6\alpha_2 + 4\alpha_3 + 2\alpha_4 \\ \varpi_3 &= 4\alpha_1 + 8\alpha_2 + 6\alpha_3 + 3\alpha_4 \\ \varpi_4 &= 2\alpha_1 + 4\alpha_2 + 3\alpha_3 + 2\alpha_4 \end{aligned}$$

Observe that  $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = \varpi_1 - \varpi_3 + \varpi_4$ , and hence not a dominant integral weight of  $F_4$ .

The underlying vector space of the exceptional Lie algebra  $E_6$  is  $V = \{v \in \mathbb{R}^8 | (v, e_6 - e_7) = (v, e_7 + e_8) = 0\}$ , and the root system is given by  $\Phi = \{\pm e_i \pm e_j | i < j \leq 5\} \cup \{\frac{1}{2} \sum_{i=1}^8 (-1)^{n(i)} e_i \in V | \sum_{i=1}^8 n(i) \text{ even}\}$ . The set of simple roots is  $\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$ , where  $\alpha_1 = \frac{1}{2}(e_8 - e_7 - e_6 - e_5 - e_4 - e_3 - e_2 + e_1)$ ,  $\alpha_2 = e_2 + e_1$ ,  $\alpha_3 = e_2 - e_1$ ,  $\alpha_4 = e_3 - e_2$ ,

$\alpha_5 = e_4 - e_3$ , and  $\alpha_6 = e_5 - e_4$ . The fundamental weights, in terms of the simple roots, are

$$\begin{aligned}\varpi_1 &= \frac{1}{3}(4\alpha_1 + 3\alpha_2 + 5\alpha_3 + 6\alpha_4 + 4\alpha_5 + 2\alpha_6) \\ \varpi_2 &= \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 \\ \varpi_3 &= \frac{1}{3}(5\alpha_1 + 6\alpha_2 + 10\alpha_3 + 12\alpha_4 + 8\alpha_5 + 4\alpha_6) \\ \varpi_4 &= 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 4\alpha_5 + 2\alpha_6 \\ \varpi_5 &= \frac{1}{3}(4\alpha_1 + 6\alpha_2 + 8\alpha_3 + 12\alpha_4 + 10\alpha_5 + 5\alpha_6) \\ \varpi_6 &= \frac{1}{3}(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6)\end{aligned}$$

Observe that  $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 = \varpi_1 + \varpi_2 - \varpi_4 + \varpi_6$ , and hence not a dominant integral weight of  $E_6$ .

The underlying vector space of the exceptional Lie algebra  $E_7$  is  $V = \{v \in \mathbb{R}^8 | (v, e_7 + e_8) = 0\}$ , and the root system is given by  $\Phi = \{\pm e_i \pm e_j | i < j \leq 6\} \cup \{\pm(e_7 - e_8)\} \cup \{\frac{1}{2} \sum_{i=1}^8 (-1)^{n(i)} e_i \in V | \sum_{i=1}^8 n(i) \text{ even}\}$ . The set of simple roots is  $\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$ , where  $\alpha_1 = \frac{1}{2}(e_8 - e_7 - e_6 - e_5 - e_4 - e_3 - e_2 + e_1)$ ,  $\alpha_2 = e_2 + e_1$ ,  $\alpha_3 = e_2 - e_1$ ,  $\alpha_4 = e_3 - e_2$ ,  $\alpha_5 = e_4 - e_3$ ,  $\alpha_6 = e_5 - e_4$ , and  $\alpha_7 = e_6 - e_5$ . The fundamental weights, in terms of the simple roots, are

$$\begin{aligned}\varpi_1 &= 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 \\ \varpi_2 &= \frac{1}{2}(4\alpha_1 + 7\alpha_2 + 8\alpha_3 + 12\alpha_4 + 9\alpha_5 + 6\alpha_6 + 3\alpha_7) \\ \varpi_3 &= 3\alpha_1 + 4\alpha_2 + 6\alpha_3 + 8\alpha_4 + 6\alpha_5 + 4\alpha_6 + 2\alpha_7 \\ \varpi_4 &= 4\alpha_1 + 6\alpha_2 + 8\alpha_3 + 12\alpha_4 + 9\alpha_5 + 6\alpha_6 + 3\alpha_7 \\ \varpi_5 &= \frac{1}{2}(6\alpha_1 + 9\alpha_2 + 12\alpha_3 + 18\alpha_4 + 15\alpha_5 + 10\alpha_6 + 5\alpha_7) \\ \varpi_6 &= 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 2\alpha_7 \\ \varpi_7 &= \frac{1}{2}(2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7)\end{aligned}$$

Observe that  $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 = \varpi_1 + \varpi_2 - \varpi_4 + \varpi_7$ , and hence not a dominant integral weight of  $E_7$ .

The underlying vector space of the exceptional Lie algebra  $E_8$  is  $V = \mathbb{R}^8$ , and the root system is given by  $\Phi = \{\pm e_i \pm e_j | i < j\} \cup \{\frac{1}{2} \sum_{i=1}^8 (-1)^{n(i)} e_i | \sum_{i=1}^8 n(i) \text{ even}\}$ . The set of simple roots is  $\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8\}$ , where  $\alpha_1 = \frac{1}{2}(e_8 - e_7 - e_6 - e_5 - e_4 - e_3 - e_2 + e_1)$ ,  $\alpha_2 = e_2 + e_1$ ,  $\alpha_3 = e_2 - e_1$ ,  $\alpha_4 = e_3 - e_2$ ,  $\alpha_5 = e_4 - e_3$ ,  $\alpha_6 = e_5 - e_4$ ,  $\alpha_7 = e_6 - e_5$ , and  $\alpha_8 = e_7 - e_6$ .



and  $\alpha_8 = e_7 - e_6$ . The fundamental weights, in terms of the simple roots, are

$$\begin{aligned}\varpi_1 &= 4\alpha_1 + 5\alpha_2 + 7\alpha_3 + 10\alpha_4 + 8\alpha_5 + 6\alpha_6 + 4\alpha_7 + 2\alpha_8 \\ \varpi_2 &= 5\alpha_1 + 8\alpha_2 + 10\alpha_3 + 15\alpha_4 + 12\alpha_5 + 9\alpha_6 + 6\alpha_7 + 3\alpha_8 \\ \varpi_3 &= 7\alpha_1 + 10\alpha_2 + 14\alpha_3 + 20\alpha_4 + 16\alpha_5 + 12\alpha_6 + 8\alpha_7 + 4\alpha_8 \\ \varpi_4 &= 10\alpha_1 + 15\alpha_2 + 20\alpha_3 + 30\alpha_4 + 24\alpha_5 + 18\alpha_6 + 12\alpha_7 + 6\alpha_8 \\ \varpi_5 &= 8\alpha_1 + 12\alpha_2 + 16\alpha_3 + 24\alpha_4 + 20\alpha_5 + 15\alpha_6 + 10\alpha_7 + 5\alpha_8 \\ \varpi_6 &= 6\alpha_1 + 9\alpha_2 + 12\alpha_3 + 18\alpha_4 + 15\alpha_5 + 12\alpha_6 + 8\alpha_7 + 4\alpha_8 \\ \varpi_7 &= 4\alpha_1 + 6\alpha_2 + 8\alpha_3 + 12\alpha_4 + 10\alpha_5 + 8\alpha_6 + 6\alpha_7 + 3\alpha_8 \\ \varpi_8 &= 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8\end{aligned}$$

Observe that  $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8 = \varpi_1 + \varpi_2 - \varpi_4 + \varpi_8$ , and hence not a dominant integral weight of  $E_8$ .  $\square$

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